Limit theorems for radial random walks on Euclidean spaces of high dimensions

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Abstract

Let $\nu \in M^1([0, \infty[)$ be a fixed probability measure. For each dimension $p \in \mathbb{N}$, let $(X_n^p)_{n \geq 1}$ be i.i.d. $\mathbb{R}^p$-valued random variables with radially symmetric distributions and radial distribution ν. We investigate the distribution of the Euclidean length of $S_n^p := X_1^p + \ldots + X_n^p$ for large parameters n and p. Depending on the growth of the dimension $p = p_n$ we derive by the method of moments two complementary CLT's for the functional $\|S_n^p\|_2$ with normal limits, namely for $n/p_n \to \infty$ and $n/p_n \to 0$. Moreover, we present a CLT for the case $n/p_n \to c \in]0, \infty[$. Thereby we derive explicit formulas and asymptotic results for moments of radial distributed random variables on $\mathbb{R}^p$.

All limit theorems are considered also for orthogonal invariant random walks on the space $M_{p,q}(\mathbb{R})$ of $p \times q$ matrices instead of $\mathbb{R}^p$ for $p \to \infty$ and some fixed dimension q.

1 Introduction

The results in this paper are motivated by the following problem: Let $\nu \in M^1([0, \infty[)$ be a fixed probability measure. Then for each dimension $p \in \mathbb{N}$ there is a unique rotation invariant probability measure $\nu_p \in M^1(\mathbb{R}^p)$ with $\varphi_p(\nu_p) = \nu$, where $\varphi_p(x) := \|x\|_2$ is the norm mapping. For each $p \in \mathbb{N}$ consider i.i.d. $\mathbb{R}^p$-valued random variables X_k^p, $k \in \mathbb{N}$, with law ν_p as well as the associated radial random walks

$$\left(S_n^p := \sum_{k=1}^n X_k^p \right)_{n \geq 0}$$

on $\mathbb{R}^p$. We are interested in finding central limit theorems for the $[0, \infty[$-valued random variables $\|S_n^p\|_2$ for $n, p \to \infty$ coupled in a suitable way. In this paper we derive the following two associated central limit theorems under disjoint growth conditions for $p = p_n$.

Theorem 1.1. *Assume that $\nu \in M^1([0, \infty[)$ admits finite moments $r_k(\nu) := \int_0^\infty x^k d\nu(x) < \infty$ for $k \leq 4$. Let $(p_n)_n$ be a sequence of dimensions with $\lim_{n \to \infty} p_n = \infty$.*

(1) If $\lim_{n \to \infty} \frac{n}{p_n} = \infty$, then

$$\frac{\sqrt{p_n}}{n}(\|S_n^{p_n}\|_2^2 - nr_2(\nu))$$

tends in distribution for $n \to \infty$ to the normal distribution $N(0, 2r_2(\nu)^2)$.

(2) If $\lim\limits_{n \to \infty} \frac{n}{p_n} = c \in [0, \infty[$, then

$$\frac{1}{\sqrt{n}}(\|S_n^{p_n}\|_2^2 - nr_2(\nu))$$

tends in distribution for $n \to \infty$ to the normal distribution $N(0, r_4(\nu) - (1 - 2c)r_2(\nu)^2)$.

Parts of this theorem were derived in [13] by using completely different methods. More precisely, CLT's above were proven for sequences $(p_n)_n$ with some strong restriction. The first CLT with the restriction $n/p_n^3 \to \infty$, was identified by M. Voit as an obvious consequence of Berry-Esseen estimates on $\underline{\mathrm{R}}^p$ with explicit constants depending on the dimension p, which are due to Bentkus and Götze [1, 2]. The proof of the second CLT with the restriction $n^2/p_n \to 0$, was derived in [13] as a consequence of asymptotic properties of so called Bessel convolutions (for a survey about the Bessel convolutions we recommend [9]).

With the approach used in [13] one is not able to get rid of the strong conditions on the growth of $p = p_n$. In particular, the mixed case $p_n = c \cdot n$ for some constant c, which builds a bridge between the CLT's with $n << p_n$ and $n >> p_n$ was stated there as an open problem.

Other associated limit theorems as laws of large numbers and large deviation principle were studied in [10]. For example, there was proven that

$$\frac{1}{n}\|S_n^{p_n}\|_2^2 \longrightarrow \int_0^\infty x^2 d\nu(x) \quad \underline{\mathrm{P}} \text{ - a.s.}$$

under the condition that p_n grows fast enough.

Theorem 1.1 will appear as special case of an extension which concerns a matrix-valued version. We consider the following geometric situation: For p, $q \in \underline{\mathrm{N}}$ we will denote by $\underline{\mathrm{M}}_{p,q}$ the space of $p \times q$-matrices over the field of real numbers $\underline{\mathrm{R}}$. Let further $\underline{\mathrm{H}}_q$ be the space of symmetric $q \times q$-matrices. Moreover, we will denote by Π_q the cone of positive semidefinite $q \times q$ matrices in $\underline{\mathrm{H}}_q$ We regard $\underline{\mathrm{M}}_{p,q}$ as a real vector space of dimension pq, equipped with the Euclidean scalar product $\langle x, y \rangle := tr(x'y)$ and norm $\|x\| = \sqrt{tr(x'x)}$ where x' is the transpose of x and tr is the trace in $\underline{\mathrm{M}}_q := \underline{\mathrm{M}}_{q,q}$. In the square case $p = q$, $\|\cdot\|$ is just the Frobenius norm. The orthogonal group $\underline{\mathrm{O}}_p$ acts on $\underline{\mathrm{M}}_{p,q}$ by left multiplication,

$$\underline{\mathrm{O}}_p \times \underline{\mathrm{M}}_{p,q} \to \underline{\mathrm{M}}_{p,q}, \quad (A, x) \mapsto Ax. \tag{1.1}$$

By uniqueness of the polar decomposition, two matrices x, $y \in \underline{\mathrm{M}}_{p,q}$ belong to the same $\underline{\mathrm{O}}_p$-orbit if and only if $x'x = y'y$. Thus the space $\underline{\mathrm{M}}_{p,q}^{\underline{\mathrm{O}}_p}$ of $\underline{\mathrm{O}}_p$-orbits in $\underline{\mathrm{M}}_{p,q}$ is naturally parameterized by the cone Π_q via the map

$$x^{\underline{\mathrm{O}}_p} \mapsto \sqrt{x'x} =: |x|, \quad \underline{\mathrm{M}}_{p,q}^{\underline{\mathrm{O}}_p} \to \Pi_q,$$

where for $r \in \Pi_q$, the matrix $\sqrt{r} \in \Pi_q$ denotes the unique positive semidefinite square root of r. According to this, the map

$$\varphi_p : \underline{\mathrm{M}}_{p,q} \to \Pi_q, \quad x \mapsto \sqrt{x'x}$$

will be regarded as the canonical projection $\underline{\mathrm{M}}_{p,q} \to \underline{\mathrm{M}}_{p,q}^{\underline{\mathrm{O}}_p}$.

In the case $q = 1$ we have $\underline{M}_{p,1} \cong \underline{R}^p$, $\underline{H}_1 = \underline{R}$, $\Pi_1 = [0, \infty[$ and φ_p is the usual norm mapping $\|\cdot\|_2 : \underline{R}^p \to [0, \infty[$. Let us now fix a parameter $q \in \underline{N}$. By taking images of measures, φ_p induces a Banach space isomorphism between the space $\underline{M}_b^{\underline{O}_p}(\underline{M}_{p,q})$ of all bounded radial (i.e. $\underline{O}_p$ invariant) Borel measures on $\underline{M}_{p,q}$ and the space $\underline{M}_b(\Pi_q)$ of bounded Borel measures on the cone Π_q. In particular, for each measure $\nu \in \underline{M}^1(\Pi_q)$ and parameter p there is a unique radial probability measure $\nu_p := \nu_{p,q} \in \underline{M}^1(\underline{M}_{p,q})$ with $\varphi_p(\nu_p) = \nu$.

Let $\nu \in \underline{M}^1(\Pi_q)$ be a fixed probability measure and $q \in \underline{N}$. As in the case $q = 1$, we now consider for each "dimension" $p \in \underline{N}$ the associated radial measures ν_p on $\underline{M}_{p,q}$ and the radial random walks $(S_n^p := \sum_{k=1}^n X_k^p)_{n \geq 0}$, i.e. X_k^p, $k \in \underline{N}$ are independent ν_p-distributed random variables.

With this notations, we shall derive the following generalization of Theorem 1.1:

Theorem 1.2. *Assume that $\nu \in \underline{M}^1(\Pi_q)$ with $\int_{\Pi_q} \|s\|^4 \, d\nu(s) < \infty$. Let $(p_n)_{n \in \underline{N}}$ be a sequence of dimensions with $\lim_{n \to \infty} p_n = \infty$.*

(1) If $\lim_{n \to \infty} \frac{n}{p_n} = \infty$, then the Π_q-valued random variable

$$\frac{\sqrt{p_n}}{n} \Big(\varphi_{p_n}^2(S_n^{p_n}) - n \int_{\Pi_q} s^2 d\nu(s) \Big)$$

tends in distribution to some normal distribution $\underline{N}(0, T^2(\nu))$ on the vector space $\underline{M}_q$ with some covariance matrix $T^2(\nu)$, wich will be described in Theorem 3.1 precisely.

(2) If $\lim_{n \to \infty} \frac{n}{p_n} = c \in [0, \infty[$, then the Π_q-valued random variable

$$\frac{1}{\sqrt{n}} \Big(\varphi_{p_n}^2(S_n^{p_n}) - n \int_{\Pi_q} s^2 d\nu(s) \Big)$$

tends in distribution to the normal distribution $\underline{N}(0, \Sigma^2(\nu) + cT^2(\nu))$ on the vector space $\underline{M}_q$ where $\Sigma^2(\nu)$ is the covariance matrix of the Π_q-valued random variable $\varphi_{p_n}^2(X_1^{p_n})$. Note that $\Sigma^2(\nu)$ depends only on ν and is independent of p_n.

We shall derive Theorem 1.2 in this higher rank setting in Section 3. The proof will rely on asymptotic results for moment functions of so called radial distributed random variables on $\underline{M}_{p,q}$ for $p \to \infty$ as well as on some identities for matrix variate normal distributions.

The organization of the paper is a follows: In section 2, some preliminaries for the proof of the main result 1.2 are presented. More precisely, in Subsection 2.1, after recalling some basic facts about relevant matrix algebra we derive a generalization of so called permutation equivalence property for Kronecker products. In 2.2 we generalize the multinomial theorem for non commutative operations. In Subsection 2.3, background on Bessel functions on the cone Π_q is provided. Subsections 2.4-2.6 are devoted to the study on the moments of radial measures and of matrix variate normal distributions respectively. In Section 3 our main result is formulated and proved.

2 Preliminaries

2.1 Kronecker and Hadamard products

In this section we collect some known facts about Kronecker and Hadamard products. The material is taken from [7].

Let $\otimes$ denotes the *Kronecker product* over the field of real numbers $\underline{\mathrm{R}}$, that is, $\otimes$ is an operation on two matrices of arbitrary size over $\underline{\mathrm{R}}$ resulting in a block matrix. It gives the matrix of the tensor product with respect to a standard choice of basis. With that the Kronecker product of $A = [a_{ij}] \in \underline{\mathrm{M}}_{m,n}$ and $B = [b_{ij}] \in \underline{\mathrm{M}}_{p,q}$ is the block matrix

$$A \otimes B := [a_{ij}B] \in \underline{\mathrm{M}}_{mp,nq}.$$

The Kronecker product is bilinear and associative but not commutative. However, $A \otimes B$ and $B \otimes A$ are *permutation equivalent*, meaning that there exist permutation matrices P and Q such that

$$A \otimes B = P \cdot (B \otimes A) \cdot Q. \tag{2.1}$$

If A and B are square matrices, then $A \otimes B$ and $B \otimes A$ are even *permutation similar*, meaning that we can take $P = Q'$. If A, B, C and D are matrices of such size that one can form the matrix products $A \cdot C$ and $B \cdot D$, then

$$(A \otimes B) \cdot (C \otimes D) = A \cdot C \otimes B \cdot D. \tag{2.2}$$

This is called the *mixed-product property*, because it mixes the ordinary matrix product and the Kronecker product. If two matrices P and Q are permutation, orthogonal or positive definite matrices then so is also the Kronecker product $P \otimes Q$.

The k-th Kronecker power $A^{\otimes k}$ is defined inductively for all positive integers k by

$$A^{\otimes 1} = A \quad \text{and} \quad A^{\otimes k} = A \otimes A^{\otimes(k-1)} \quad \text{for } k = 2, 3, \dots.$$

This definition implies that for $A \in \underline{\mathrm{M}}_{m,n}$, we have $A^{\otimes k} \in \underline{\mathrm{M}}_{m^k,n^k}$.

For a matrix $X \in \underline{\mathrm{M}}_{m,n}$, $vec(X)$ is the $m \cdot n \times 1$ vector defined as

$$vec(X) = (x_1', \dots, x_m')' \in \underline{\mathrm{M}}_{m \cdot n, 1},$$

where x_i, $i = 1, \dots, n$ is the i-th column of X.

We now derive a generalization of permutation equivalence property, which will be required for the proof of Theorem 3.1 below.

Lemma 2.1. *Let $A_i \in \underline{M}_{p_i,q_i}$ $(i = 1, \dots, k)$, $p := p_1 \cdot \ldots \cdot p_k$ and $q := q_1 \cdot \ldots \cdot q_k$. Then, for each permutation $\sigma \in Sym(\{1, \dots, k\})$ there exist permutation matrices $P_\sigma \in \underline{M}_{p,p}$ and $Q_\sigma \in \underline{M}_{q,q}$ such that*

$$A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(k)} = P_\sigma \cdot (A_1 \otimes \ldots \otimes A_k) \cdot Q_\sigma.$$

Proof. Without loss of generality we can assume that $k = 4$, for the Kronecker product is associative. Since $(1) \otimes M = M = M \otimes (1)$ for any matrices M, it suffices to show that $A_1 \otimes A_3 \otimes A_2 \otimes A_4$ is permutation equivalent to $A_1 \otimes A_2 \otimes A_3 \otimes A_4$. For a matrix M let I_M and I^M denote the identity matrices of such size that one can form the matrix products $I_M \cdot M$ and $M \cdot I^M$. By the property (2.1) there exist permutation matrices P and Q with $A_3 \otimes A_2 = P(A_2 \otimes A_3)Q$. Therefore, using (2.2) we obtain by an easy computation

$$A_1 \otimes A_3 \otimes A_2 \otimes A_4 = (I_{A_1} \otimes P \otimes I_{A_4}) \cdot (A_1 \otimes A_2 \otimes A_3 \otimes A_4) \cdot \left(I^{A_1} \otimes Q \otimes I^{A_4} \right).$$

Clearly, both $I_{A_1} \otimes P \otimes I_{A_4}$ and $I^{A_1} \otimes Q \otimes I^{A_4}$ are permutation matrices. This completes the proof. $\square$

In the following, let $A = [a_{ij}]$, $B = [b_{ij}] \in \underline{\mathrm{M}}_{p,q}$ of the same dimensions. The *Hadamard product*, also known as the entrywise product of A and B is denoted by $A \circ B$ and is defined to be the matrix

$$A \circ B := [a_{ij} b_{ij}] \in \underline{\mathrm{M}}_{p,q}.$$

The Hadamard product is commutative, associative and distributive w.r.t. addition, and is a principal submatrix of the Kronecker product.

For a matrix M, let us denote by $\mathbb{1}_M$ the 1-matrix of the same dimension as M, that is, $\mathbb{1}_M = (c_{ij})_{ij}$ with $c_{ij} = 1$ for all i, j. We will write it simply $\mathbb{1}$ when no confusion will arise. It is clear that

$$A \otimes B = (A \otimes \mathbb{1}) \circ (\mathbb{1} \otimes B), \tag{2.3}$$

$$B \otimes A = (\mathbb{1} \otimes A) \circ (B \otimes \mathbb{1}). \tag{2.4}$$

Let P and Q be permutation matrices of such size that one can form the matrix products $P \cdot A$ and $A \cdot Q$. It is easy to check that

$$P(A \circ B)Q = (PAQ) \circ (PBQ). \tag{2.5}$$

2.2   Permutations on a multiset

In this section, we generalize the multinomial theorem in terms of Kronecker product instead of the usual multiplication. In order to do this, we first recall the notion of the permutation on a multiset from [11, Chapter 1].

Let $u \in \underline{\mathrm{N}}$ and $k \in \underline{\mathrm{N}}_0$. We denote by $C_0(k,u)$ the set of all u-compositions of k, that is,

$$C_0(k,u) = \left\{ \lambda \in \underline{\mathrm{N}}_0^u : \quad |\lambda| := \sum_{i=1}^{u} \lambda_i = k \right\},$$

and write $C(k,u)$ instead of $C_0(k,u) \cap \underline{\mathrm{N}}^u$. Moreover, we set $M_u := \{1, 2, \dots, u\}$. For a $\lambda \in C(k,u)$ a *finite multiset $Mult(\lambda)$* on the ordered set M_u is a is a set, where i is contained with the multiplicity λ_i for all $i \in M_u$. One regards λ_i as the number of repetitions of i. A permutation $\pi = (\pi_1 \pi_2 \dots \pi_k)$ on $Mult(\lambda)$ can be defined as a linear ordering of the elements of $Mult(\lambda)$, that is, an element $i \in M$ appears exactly λ_i times in the permutation π. The set of all permutation on $Mult(\lambda)$ will be denoted by $\mathfrak{S}(\lambda)$. A permutation $\pi = (\pi_1 \pi_2 \cdots \pi_k)$ on $Mult(\lambda; M)$ can be regarded as a way to place k distinguishable balls in u distinguishable boxes such that the i-th box contains λ_i balls. Indeed, if i $(i = 1, \dots, u)$ appears in position $j \in \{1, \dots, k\}$ of the permutation π, then we put the "ball" π_j into the box i. For instance let $u = 3$, $\lambda := (1, 3, 2) \in C(k, u)$ be a 3-composition of $k = 6$ and $\pi = (2\ 1\ 2\ 3\ 3\ 2) =:$ $(\pi_1\ \pi_2 \dots \pi_6)$ be a permutation on $Mult(\lambda)$ then we put π_2 in the first box, π_1, π_3, π_6 in the second box and π_4, π_5 in the third box. It is clear that

$$|\mathfrak{S}(\lambda)| = \binom{k}{\lambda_1, \dots, \lambda_u} := \frac{k!}{\lambda_1! \dots \lambda_u!}.$$

Let $m_i \in \underline{M}_{p_i,q_i}$ $(i = 1, \ldots, u)$, $\lambda \in C(k, u)$ and $\pi = (\pi_1, \ldots, \pi_k) \in \mathfrak{S}(\lambda)$. We will write $\pi(m_1, \ldots, m_u)$ instead of $m_{\pi_1} \otimes m_{\pi_2} \otimes \cdots \otimes m_{\pi_k}$. Moreover, we set

$$W(n, u) := \left\{ \mu = (\mu_1, \ldots, \mu_u) \in \{1, \ldots, n\}^u : \mu_1 < \mu_2 < \cdots < \mu_u \right\}.$$

In the following theorem, which will be used in Section 3 several times, we expand a Kronecker power of a matrix sum in terms of powers of the terms in that sum.

Theorem 2.2. *Let $k \in \underline{N}$ and $x_1, \ldots, x_n \in \underline{M}_{p,q}$. Then*

$$\left(\sum_{i=1}^{n} x_i \right)^{\otimes, k} = \sum_{u=1}^{k} \sum_{\lambda \in C(k,u)} \sum_{\mu \in W(n,u)} \sum_{\pi \in \mathfrak{S}(\lambda)} \pi(x_{\mu_1}, \ldots, x_{\mu_u}). \tag{2.6}$$

For $p = q = 1$ the Kronecker product coincides with the usual multiplication on $\underline{R}$ and therefore, (2.6) generalizes multinomial formula. For indices $u \in \{1, \ldots, k\}$, $\mu = (\mu_1, \ldots, \mu_n) \in W(n, u)$, $\lambda \in C(k, u)$ and $\pi \in \mathfrak{S}(\lambda)$ let us consider the associated summand

$$\pi(x_{\mu_1}, \ldots, x_{\mu_u}) = x_{\mu_{\pi_1}} \otimes \ldots \otimes x_{\mu_{\pi_k}} \tag{2.7}$$

from (2.6). It is clear that the different matrices $x_{\mu_1}, \ldots, x_{\mu_u}$, the numbers of their repetitions and their exact positions in the Kronecker product (2.7) are described by $\mu = (\mu_1, \ldots, \mu_u) \in W(n, u)$, $\lambda = (\lambda_1, \ldots, \lambda_u) \in C(k, u)$ and $\pi = (\pi_1, \ldots, \pi_k) \in \mathfrak{S}(\lambda)$ respectively.

Proof. We proceed by induction on k. For $k = 1$ there is nothing to proof. Next suppose as induction hypothesis that (2.6) holds with $k - 1$ instead of k. It gives

$$\left(\sum_{i=1}^{n} x_i \right)^{\otimes, k} = \sum_{u=1}^{k-1} \sum_{\lambda \in C(k-1,u)} \sum_{\mu \in W(n,u)} \sum_{\pi \in \mathfrak{S}(\lambda)} \pi(x_{\mu_1}, \ldots, x_{\mu_u}) \otimes \sum_{j=1}^{n} x_j$$

$$= \sum_{j=1}^{n} \sum_{u=1}^{k-1} \sum_{\lambda \in C(k-1,u)} \sum_{\mu \in W(n,u)} \sum_{\pi \in \mathfrak{S}(\lambda)} \pi(x_{\mu_1}, \ldots, x_{\mu_u}) \otimes x_j. \tag{2.8}$$

Consider a term $\pi(x_{\mu_1}, \ldots, x_{\mu_u}) \otimes x_j$ of the sum above, that is, $j \in \{1, \ldots, n\}$, $u \in \{1, \ldots, k-1\}$, $\lambda \in C(k-1, u)$, $\mu \in W(n, u)$ and $\pi \in \mathfrak{S}(\lambda)$. If there is $\beta \in \{1, \ldots, u\}$ with $j = \mu_\beta$ then it corresponds to exact one summand in (2.6) associated with indices $\tilde{u} = u$, $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{\beta-1}, \lambda_\beta + 1, \lambda_{\beta+1}, \ldots, \lambda_u)$, $\tilde{\mu} = \mu$ and $\tilde{\pi} = (\pi_1, \ldots, \pi_{k-1}, \beta)$. In the other case, that is, if $j \in (\mu_{\beta-1}, \mu_\beta)$ for an $\beta \in \{1, \ldots, u+1\}$ with the convention $\mu_0 := 0$ and $\mu_{u+1} = \infty$ the term $\pi(x_{\mu_1}, \ldots, x_{\mu_u}) \otimes x_j$ corresponds to a summand in (2.6) associated with indices $\tilde{u} = u+1$, $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{\beta-1}, 1, \lambda_\beta, \ldots, \lambda_u)$, $\tilde{\mu} = (\mu_1, \ldots, \mu_{\beta-1}, j, \mu_\beta, \ldots, \mu_u)$ and $\tilde{\pi} = (\pi_1, \ldots, \pi_{k-1}, \beta)$. As the number of summands in both (2.6) and (2.8) is equal to n^k, the induction step follows. $\square$

In the following we collect some known facts about multivariate Bessel functions on the cone Π_q, which will be needed later. The material is mainly taken from [9]. We also refer to [4] and [6].

2.3 Bessel functions on the cone Π_q

Let Z_λ denote the *zonal polynomials*, which are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q) \in \mathbb{N}_0^q$ (we write $\lambda \geq 0$ for short) and normalized such that

$$tr(x)^k = \sum_{|\lambda|=k} Z_\lambda(x) \quad \forall \, k \in \mathbb{N}_0;$$

see [4] for the construction of Z_λ and further details. It is well known that the Z_λ are homogeneous polynomials which are invariant under conjugation by O_q and thus depend only on the eigenvalues of their argument. More precisely, for $x \in H_q$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$, one has

$$Z_\lambda(x) = C_\lambda^\alpha(\xi) \quad \text{with} \quad \alpha = 2$$

where the C_λ^α are the *Jack polynomials* of index α in a suitable normalization (see [4],[9]). The Jack polynomials C_λ^α are homogeneous of degree $|\lambda|$ and symmetric in their arguments. Let $\alpha > 0$ be a fixed parameter. For partitions $\lambda = (\lambda_1, \ldots, \lambda_q)$ we introduce the *generalized Pochhammer symbol*

$$(\mu)_\lambda^\alpha = \prod_{j=1}^q \left(\mu - \frac{1}{\alpha}(j-1) \right)_{\lambda_j} \quad (\mu \in \mathbb{C}),$$

where $(\cdot)_j$ denotes the usual Pochhammer symbol. For an index $\mu \in \mathbb{C}$ satisfying $(\mu)_\lambda^\alpha \neq 0$ for all $\lambda \geq 0$ the *matrix Bessel functions* associated with the cone Π_q are defined as ${}_0F_1$-hypergeometric series in terms of the Z_λ, namely

$$J_\mu(x) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda^{d/2}|\lambda|!} Z_\lambda(x). \tag{2.9}$$

For a general background on matrix Bessel functions, the reader is referred to the fundamental article [6]. If $q = 1$, then $\Pi_q = [0, \infty[$ and we have $J_\mu(x^2/4) = j_{\mu-1}(x)$, where $j_\kappa(z) = {}_0F_1(\kappa + 1; -z^2/4)$ is the modified Bessel function in one variable.

2.4 Polynomials on $M_{p,q}$

Let $p, q \in \mathbb{N}$. For $\kappa = (\kappa_{ij})_{i,j} \in \mathbb{N}_0^{p \times q}$ (a composition) we set $|\kappa| := \sum_{i,j} \kappa_{ij}$ and $R_i(\kappa) := \sum_{j=1}^q \kappa_{ij}$, $i = 1, \ldots, p$. Moreover, we write $z^\kappa := \prod_{i,j} z_{ij}^{\kappa_{ij}}$. Clearly, z^κ is a monomial of degree $|\kappa|$. The spaces of *polynomials* and *row-even polynomials* are defined by

$$P := span\Big\{ x^\kappa : \ \kappa \in \mathbb{N}_0^{p \times q} \Big\},$$

$$P_e := span\Big\{ x^\kappa : \ \kappa \in \mathbb{N}_0^{p \times q}, \forall \, i \ R(i) \text{ is even} \Big\}$$

respectively.

We shall need the following observation:

Lemma 2.3. *Let $r \in \Pi_q$, and $\kappa \in \mathbb{N}_0^{p \times q}$. Then*

$$\Psi_{r,\kappa} : M_{p,q} \to \mathbb{R}, \quad \Psi_{r,\kappa}(z) := ((zr)'(zr))^\kappa$$

is an even polynomial of degree $2|\kappa|$.

Proof. Since the product of two row-even polynomials is also a row-even polynomial, the proof follows easily by induction on $n = |\kappa|$. $\qquad\square$

2.5 Radial measures on $\underline{\mathrm{M}}_{p,q}$ and their moments

In this section we study radial measures on the space $\underline{\mathrm{M}}_{p,q}$. In particular, we derive asymptotic results for their moments as $p \to \infty$. This results will play a key role in the proof of Theorem 3.1. We start with the definition of a radial measure on $\underline{\mathrm{M}}_{p,q}$.

Definition 2.4. A measure ν_p on $\underline{\mathrm{M}}_{p,q}$ is called *radial* if

$$A(\nu_p) = \nu_p \quad \forall \; A \in \underline{\mathrm{O}}_p,$$

that is, if it is invariant under the action (1.1). In particular, for $q = 1$ a measure ν_p on $\underline{\mathrm{R}}^p$ is radial if it is invariant under rotations.

Remark 2.5. It is well known that for each probability measure $\nu \in \underline{\mathrm{M}}^1(\Pi_q)$ and a dimension $p \in \underline{\mathrm{N}}$ there is a unique radial probability measure $\nu_p \in \underline{\mathrm{M}}^1(\underline{\mathrm{M}}_{p,q})$ with ν as its radial part, that is, $\varphi_p(\nu_p) = \nu$.

In order to study radial measures on $\underline{\mathrm{M}}_{p,q}$ and their moments we need an analogue of a sphere in our higher rank setting. For an $r \in \Pi_q$ we define a sphere of radius r as the set

$$\Sigma_{p,q}^r = \left\{ x \in \underline{\mathrm{M}}_{p,q} : \; \sqrt{x'x} = r \right\}.$$

Clearly, $\Sigma_{p,q}^r$ is the orbit of the block matrix $\sigma_r := (r\ 0)' \in \underline{\mathrm{M}}_{p,q}$ according to the operation (1.1). For simplicity of notation, we write $\Sigma_{p,q}$ instead of $\Sigma_{p,q}^{I_q}$, where $I_q \in \underline{\mathrm{R}}^{q \times q}$ denotes the identity matrix. In the case $q = 1$ we identify $\Sigma_{p,1}^r$ with the Euclidean sphere of radius $r \in [0, \infty[$. Moreover, let us denote by U_p^r the uniform distribution on a sphere $\Sigma_{p,q}^r$.

One can easily show that a radial probability measure ν_p with its radial part $\nu \in \underline{\mathrm{M}}^1(\Pi_q)$ enables the decomposition

$$\nu_p(\cdot) = \int_{\underline{\mathrm{M}}_{p,q}} U_p^{\varphi_p(x)}(\cdot) d\nu_p(x) = \int_{\Pi_q} U_p^r(\cdot) d\nu(r) \in \underline{\mathrm{M}}^1(\underline{\mathrm{M}}_{p,q}). \tag{2.10}$$

In the sense of Jewett [8], the formula above is an example of a decomposition of a measure (here ν_p) according to so called orbital morphism (here φ_p). More precisely, φ_p is an orbital mapping, that is a proper and open continuous surjection from $\underline{\mathrm{M}}_{p,q}$ onto Π_q. The mapping $r \mapsto U_p^r$ from Π_q to $\underline{\mathrm{M}}^1(\underline{\mathrm{M}}_{p,q})$ is a recomposition of φ_p which means that each U_p^r is a probability measure on $\underline{\mathrm{M}}_{p,q}$ with support equal to $\varphi_p^{-1}(r)$ (here $= \Sigma_{p,q}^r$), and such that $\nu_p = \int_{\underline{\mathrm{M}}_{p,q}} U_p^{\varphi_p(x)} d\nu_p(x)$.

Definition 2.6. Let Z be a $\underline{\mathrm{M}}_{p,q}$-valued random variable with distribution $\mu \in \underline{\mathrm{M}}^1(\underline{\mathrm{M}}_{p,q})$. We say that $\mu \in \underline{\mathrm{M}}^1(\underline{\mathrm{M}}_{p,q})$ (or Z) admits a k-th moment ($k \in \underline{\mathrm{N}}_0$) if $\int_{\underline{\mathrm{M}}_{p,q}} \|z\|^k d\mu(z) < \infty$, and define in this case the *k-th moment* of μ (or Z) by

$$M_k(\mu) \doteq M_k(Z) \doteq \underline{\mathrm{E}}\left(Z^{\otimes, k} \right) \in \underline{\mathrm{M}}_{p^k, q^k}.$$

Let $I = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ with $i_\alpha \in \{1, \ldots, p\}$ and $j_\alpha \in \{1, \ldots, q\}$ for $\alpha \in \{1, \ldots, k\}$. Then the I-th component $M_k(Z)_I$ of $M_k(Z)$ is given by

$$M_k(Z)_I = \underline{\mathrm{E}}(Z_{i_1, j_1} \cdot \ldots \cdot Z_{i_k, j_k}).$$

Moreover, for an $\kappa \in \underline{N}_0^{p \times q}$ with $|\kappa| = k$ we set

$$m_\kappa(\mu) := \int_{\underline{M}_{p,q}} z^\kappa d\mu(z) \in \underline{R},$$

and call $m_\kappa(\mu)$ also the κ-th moment of μ.

In the following $\widehat{\mu}$ denote the characteristic function of a probability measure μ on $\underline{M}_{p,q}$, that is,

$$\widehat{\mu}(x) = \int_{\underline{M}_{p,q}} \exp(i\langle x, y \rangle) d\mu(y).$$

Let $k \in \underline{N}_0$ and $\kappa \in \underline{N}_0^{p \times q}$ with $|\kappa| = k$. If μ admits a k-th moment then we have

$$m_\kappa(\mu) = (-i)^{|\kappa|} D_\kappa \widehat{\mu}(x)|_{x=0}, \tag{2.11}$$

where D_κ is the differential operator $\frac{\partial^{\kappa_{ij}}}{\partial x_{11}^{\kappa_{11}}} \frac{\partial^{\kappa_{12}}}{\partial x_{12}^{\kappa_{12}}} \cdots \frac{\partial^{\kappa_{pq}}}{\partial x_{pq}^{\kappa_{pq}}}$.

Here and subsequently, ν_p denotes a radial probability measure on $\underline{M}_{p,q}$ with the corresponding radial part $\nu \in \underline{M}^1(\Pi_q)$ and X is a $\underline{M}_{p,q}$-valued random variable with radial distribution ν_p.

In the next lemmas we explore the covariance structure of X and compute the asymptotic behaviour of the moments of ν_p for large dimensions p.

Lemma 2.7. *Let $X = (X_{ij})_{i,j}$ be $\underline{M}_{p,q}$-valued random variable with radial distribution $\nu_p \in \underline{M}^1(\underline{M}_{p,q})$. Then*

$$\underline{E}(X) = \mathbf{0} \quad and \quad \underline{E}(X_{ji}X_{lk}) = \delta_{j,l}\underline{E}(X_{1i}X_{1k}) \tag{2.12}$$

Proof. For $r \in \underline{R}\backslash\{0\}$ let $M_{j,r}$ and $S_{i,j}$ be $p \times p$ matrices produced by multiplying all elements of row j of the identity matrix by r and by exchanging row i and row j of the identity matrix respectively. As $S_{i,j}$ is a symmetric involution on $\underline{M}_p$, we have $S_{i,j} \in \underline{O}_p$. For $r = \pm 1$ the matrix $M_{j,r}$ is also orthogonal. By assumption, X and AX are identically distributed for any $A \in \underline{O}_p$. Therefore, we have

$$\underline{E}(X)_{ij} = \underline{E}(M_{j,-1}X)_{ij} = -\underline{E}(X)_{ij}.$$

So the first equality in (2.12) holds.

Choose $i, k \in \{1, \ldots, q\}$ and $j, l \in \{1, \ldots, p\}$ with $j \neq l$. We conclude from

$$\underline{E}(X_{ji}X_{lk}) = \underline{E}((M_{j,-1}X)_{ji}(M_{j,-1}X)_{lk}) = -E(X_{ji}X_{lk})$$

that $\underline{E}(X_{ji}X_{lk}) = 0$. We now turn to the case $j = l$. The transformation $\underline{M}_{p,q} \to \underline{M}_{p,q}$, $A \mapsto S_{i,j}A$, switches all matrix elements on row i with their counterparts on row j. Therefore, from radiality of $P_X = \nu_p$ it follows that

$$\underline{E}(X_{ji}X_{jk}) = \underline{E}\left((S_{j,1}X)_{ji}(S_{j,1}X)_{jk}\right) = \underline{E}(X_{1i}X_{1k}) \quad \text{for } i, k \in \{1, \ldots, q\}.$$

$\square$

Now let us denote by x_i the i-th row of X. According to the lemma above, we have

$$\underline{C}ov(x_i, x_j) = \delta_{i,j} \cdot \underline{E}(x_1 x_1') =: T_p \in \underline{M}_q.$$

Therefore, we obtain

$$\underline{C}ov(X) := \underline{C}ov(vec(X')) = I_p \otimes T_p \in \underline{M}_{q \cdot p}.$$

Lemma 2.8. *The characteristic function for the uniform distribution U_p^r on the sphere $\Sigma_{p,q}^r$ of radius $r \in \Pi_q$ is given by*

$$\widehat{U_p^r}(z) = J_\mu \left(\frac{1}{4}(zr)'(zr) \right), \quad (z \in \underline{M}_{p,q}) \tag{2.13}$$

where $\mu = \frac{p}{2}$ and J_μ is the Bessel function of index μ of Eq. (2.9).

Proof. Let $r \in \Pi_q$. Consider the map

$$T_r : \Sigma_{p,q} \to \Sigma_{p,q}^r, \quad y \mapsto yr.$$

Since $T_r(U_p^{I_q}) = U_p^r$, we get by substitution formula

$$\widehat{U_p^r}(z) = \int_{\underline{M}_{p,q}} e^{i\langle z,y\rangle} dU_p^r(y) = \int_{\Sigma_{p,q}} e^{i\langle z,yr\rangle} dU_p^{I_q}(y).$$

On the other side, according to Proposition XVI.2.3. of [4] we have for $x \in \underline{M}_{p,q}$ the identity

$$\int_{\Sigma_{p,q}} e^{i\langle y,x\rangle} dU_p^{I_q}(y) = J_\mu \left(\frac{1}{4} x'x \right), \quad \mu = \frac{p}{2}.$$

By taking these two identities above into account, (2.13) follows as claimed. $\square$

Lemma 2.9. *Let $\kappa \in \underline{N}_0^{p\times q}$, $l := |\kappa|/2$ and $\mu = \frac{p}{2}$. The κ-th moment $m_\kappa(U_p^r)$ of the uniform distribution on $\Sigma_{p,q}^r$ is given as follows:*

(a) *If $R_i(\kappa) = \sum_{j=1}^q \kappa_{ij}$ is even for all $i = 1, \ldots, p$, then $l \in \underline{N}_0$ and*

$$m_\kappa(U_p^r) = \frac{1}{4^l \, |\kappa|!} \sum_{\lambda \in C_0(l,q)} \frac{1}{(\mu)_\lambda^{d/2}} D_\kappa \Big(Z_\lambda((zr)^*(zr)) \Big) \Big|_{z=0}. \tag{2.14}$$

(b) *If $R_i(\kappa)$ is not even for some $i = 1, \ldots, p$, then $m_\kappa(U_p^r) = 0$.*

Proof. By the Identity (2.11), the preceding lemma and (2.9) we have

$$m_\kappa(U_p^r) = (-i)^{|\kappa|} \sum_{j=0}^\infty \frac{(-1)^j}{j!} \sum_{\lambda \in C_0(j,q)} \frac{1}{(\mu)_\lambda^{d/2}} D_\kappa \left(Z_\lambda \left(\frac{1}{4}(zr)^*(zr) \right) \right) \Big|_{z=0}. \tag{2.15}$$

Let $\lambda \in \underline{N}_0^q$ and $p_r : z \mapsto Z_\lambda \left((zr)^*(zr) \right)$. Since Z_λ is a homogeneous polynomial of degree $|\lambda|$, Lemma 2.3 shows that p_r is a homogeneous, row-even polynomial of degree $2|\lambda|$. Therefore, each term on the right-hand side of (2.15) vanishes if $\kappa \in \underline{N}_0^{p\times q}$ with $R_i(\kappa)$ is odd for some $i \in \{1, \ldots, p\}$ or if $|\kappa| \neq 2|\lambda|$. This proves the assertion. $\square$

Theorem 2.10. *Let $\kappa \in \underline{N}_0^{p\times q}$, $l := |\kappa|/2$, $\nu \in \underline{M}^1(\Pi_q)$ and $\nu_p \in \underline{M}^1(\underline{M}_{p,q})$ be the corresponding radial probability measure on $\underline{M}_{p,q}$ which admits a κ-th order moment. Then the κ-th moment $m_\kappa(\nu_p)$ of ν_p exists in $\underline{R}$ and has the following asymptotic as $p \to \infty$:*

(a) *If $R_i(\kappa)$ is even for all $i = 1, \ldots, p$, then $m_\kappa(\nu_p) = O\left(\frac{1}{p^l} \right)$.*

(b) *If $R_i(\kappa)$ is not even for some $i = 1, \ldots, p$, then $m_\kappa(\nu_p) = 0$.*

Proof. The existence of $m_\kappa(\nu_p)$ is clear. By the decomposition (2.10) we obtain

$$m_\kappa(\nu_p) = \int_{\Pi_q} m_\kappa(U_p^r) d\nu(r),$$

where U_p^r is the uniform distribution on $\Sigma_{p,q}^r$. Therefore, the assertion (b) follows immediately from Lemma 2.9 (a). Now we turn to the case (a). Since the λ-th term in the sum (2.14) is a homogeneous polynomial in the variable $r_{11}, r_{12}, \ldots, r_{qq}$ of degree $2|\lambda|$ which is also independent of p, Lemma 2.9 (a) leads to

$$m_\kappa(\nu_p) = \sum_{\lambda \in C_0(l,q)} \int_{\Pi_q} O\left(\frac{1}{p^l}\right) d\nu(r) = O\left(\frac{1}{p^l}\right).$$

$\square$

2.6 Matrix variate normal distribution and their moments

In this section we derive some results concerning the class of matrix variate normal distribution on $\underline{M}_q$, to which belongs the limiting distribution in our main result 1.2.

Let $Z = (z_{ij})_{1 \le i,j \le q}$ be a real matrix variate normal distributed variable with mean matrix $\mu \in \underline{M}_q$ and symmetric covariance matrix

$$\Sigma = \left(\Sigma_{(i,j),(l,k)}\right)_{1 \le i,j,l,k \le q} = \left(\Sigma_{(l,k),(i,j)}\right)_{1 \le i,j,l,k \le q} \in \underline{M}_{q^2} \cong \underline{M}_q \otimes \underline{M}_q. \qquad (2.16)$$

We write $Z \sim \underline{N}(\mu, \Sigma)$ for short. This means that $vec(Z')$ is $\underline{N}(vec(\mu'), \Sigma)$-distributed. In order to prove some formulas for moments $M_k(Z) = \underline{E}(Z^{\otimes k})$ of Z, which we will use in Section 3, we need the following notation. Let $u \in \underline{N}$, $k := 2u$, $I = ((i_1, j_1), \ldots, (i_k, j_k)) \in (\{1, \ldots, q\}^2)^k$, $\lambda = (2, \ldots, 2) \in C(k, u)$ and $\pi = (\pi_1, \ldots, \pi_k) \in \mathfrak{S}(\lambda)$. For a tuple $v = (v_1, \ldots, v_n)$ we will write $\{v\}$ instead of the set $\{v_1, \ldots, v_n\}$. Consider the sets

$$\pi(I)_i = \{(i_\mu, j_\mu) \in \{I\} : \pi_\mu = i\} \quad (i = 1, \ldots, u).$$

Obviously $\pi(I)_i$ $(i = 1, \ldots, u)$ forms a partition of $\{I\}$ with $|\pi(I)_i| = 2$. We define for π, I and a symmetric covariance matrix Σ as in (2.16),

$$\pi(\Sigma)_I := \prod_{i=1}^u \Sigma_{(\alpha_i, \beta_i), (\gamma_i, \delta_i)} \text{ where } \{(\alpha_i, \beta_i), (\gamma_i, \delta_i)\} = \pi(I)_i.$$

For instance let $u = 2$, $I = \{(2,1), (2,2), (3,2), (2,1)\}$, $\lambda := (2,2) \in C(4,2)$ and $\pi = (1\ 2\ 1\ 2) =: (\pi_1 \ldots \pi_4)$; then we have $\pi(I)_1 = \{(2,1), (3,2)\}$, $\pi(I)_2 = \{(2,2), (2,1)\}$ and $\pi(\Sigma)_I = \Sigma_{(2,1),(3,2)} \cdot \Sigma_{(2,2),(2,1)}$.

The moment formulas $M_k(Z)$ for multivariate normal distributed random vector $Z \sim \underline{N}(\mu, \Sigma)$ are well studied in the literature (see [12] and [5]). In [12, Theorem 1] we find moment formulas for centered Gaussian distribution Z, which are derived in a relative fast and elegant way. This formula can be easily translated in our setting. Namely, the I-th component of k-th order moment of a $\underline{N}(\mathbf{0}, \Sigma)$-distributed random matrix Z is given by

$$M_k(Z)_I = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \frac{1}{u!} \sum_{\pi \in \mathfrak{S}(\lambda)} \pi(\Sigma)_I, & \text{if } k = 2u, \lambda = (2, \ldots, 2) \in C(k, u). \end{cases} \qquad (2.17)$$

In the most classical case $q = 1$, that is, Z is centered Gaussian distribution on $\underline{R}$ with variance $\sigma^2 > 0$ the identity (2.17) reduces to the well known formula

$$\underline{E}(Z^k) = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \sigma^k (k-1)(k-3) \cdot \ldots \cdot 3 \cdot 1 & \text{if } k \text{ is even.} \end{cases} \tag{2.18}$$

The following two simple observations concerning the k-th moment of normal distributed random matrix and a sum of two independent, normal distributed random matrices respectively will be needed for the proof of Theorem 3.1.

Lemma 2.11. *Let Z be $\underline{N}(\mathbf{0}, \Sigma)$-distributed random variable and $Z_1, Z_2, \ldots$ independent copies of Z. The k-th order moment of Z is given by*

$$M_k(Z) = \begin{cases} \mathbf{0}, & \text{if } k \text{ is odd,} \\ \frac{1}{u!} \sum_{\pi \in \mathfrak{S}(\lambda)} \underline{E}\pi(Z_1, \ldots, Z_u), & \text{if } k = 2u \end{cases}$$

where $\lambda = (2, \ldots, 2) \in C(2u, u)$.

Proof. Let $k \in \underline{N}$ and $I = ((i_1, j_1), \ldots, (i_k, j_k)) \in (\{1, \ldots, q\}^2)^k$. If k is odd, then it follows by (2.17) that $M_k(Z)_I = 0$. Suppose that $k = 2u$, $(u \in \underline{N})$. For $\pi \in \mathfrak{S}(\lambda)$, $\lambda = (2, \ldots, 2) \in C(k, u)$ and I as above, we have $\pi(Z_1, \ldots, Z_u)_I = (Z_{\pi_1} \otimes \ldots \otimes Z_{\pi_k})_I$. Let $\{(\alpha_i, \beta_i), (\gamma_i, \delta_i)\} = \pi(I)_i$, $i = 1, \ldots, u$. By independence it follows

$$\underline{E}\pi(Z_1, \ldots, Z_u)_I = \underline{E}(Z_{\pi_1} \otimes \ldots \otimes Z_{\pi_k})_I = \prod_{i=1}^{u} \underline{E}(Z_i \otimes Z_i)_{(\alpha_i, \beta_i),(\gamma_i, \delta_i)}$$

$$= \prod_{i=1}^{u} \Sigma_{(\alpha_i, \beta_i),(\gamma_i, \delta_i)} = \pi(\Sigma)_I.$$

The lemma is now a consequence of Eq. (2.17). $\qquad\square$

Lemma 2.12. *Let Z_i $(i = 1,\ 2)$ be independent random variables with distributions $\underline{N}(\mathbf{0}, \Sigma_i)$. Then*

$$\underline{E}\left((Z_1 + Z_2)^{\otimes, k}\right) = \sum_{l=0}^{k} \sum_{\pi \in \mathfrak{S}((l, k-l))} \underline{E}\pi(Z_1, \mathbb{1}) \circ \underline{E}\pi(\mathbb{1}, Z_2). \tag{2.19}$$

Proof. By the definition of $\circ$-product and independence of Z_1 and Z_2 we have

$$\underline{E}\left((Z_1 + Z_2)^{\otimes, k}\right) = \sum_{l=0}^{k} \sum_{\pi \in \mathfrak{S}((l, k-l))} \underline{E}\pi(Z_1, Z_2)$$

$$= \sum_{l=0}^{k} \sum_{\pi \in \mathfrak{S}((l, k-l))} \underline{E}\left(\pi(Z_1, \mathbb{1}) \circ \pi(\mathbb{1}, Z_2)\right)$$

$$= \sum_{l=0}^{k} \sum_{\pi \in \mathfrak{S}((l, k-l))} \underline{E}\pi(Z_1, \mathbb{1}) \circ \underline{E}\pi(\mathbb{1}, Z_2).$$

$\qquad\square$

3 Radial limit theorems on $\underline{M}_{p,q}$ for $p \to \infty$

Let $\nu \in \underline{M}^1(\Pi_q)$ be a fixed probability measure such that $\int_{\Pi_q} \|x\|^4 \, d\nu(x) < \infty$. Then for each dimension $p \in \underline{N}$ there is a unique radial probability measure $\nu_p \in \underline{M}^1(\underline{M}_{p,q})$ with ν as its radial part, that is, $\nu = \varphi_p(\nu_p)$. Let $X = (x_{ij})_{ij}$ be ν_p distributed random matrix on $\underline{M}_{p,q}$. We define

$$r_2(\nu) := \underline{E}\left(\varphi_p^2(X)\right) = p \cdot T_p \in \Pi_q,$$

$$\Sigma(\nu) := \underline{C}ov(\varphi_p^2(X)) = \underline{C}ov(vec(\varphi_p^2(X)')) \in \Pi_{q^2} \cong \Pi_q \otimes \Pi_q.$$

Clearly, $r_2(\nu)$ and $\Sigma(\nu)$ are independent from p. Now, we consider for each $p \in \underline{N}$ i.i.d. $\underline{M}_{p,q}$-valued random variables

$$X_k := \left(X_k^{(i,j)}\right)_{1 \le i \le p, \ 1 \le j \le q}, \quad k \in \underline{N}$$

with law ν_p as well as the random variables

$$\Xi_n^p(\nu) := \varphi_p(S_n^p)^2 - n r_2(\nu), \tag{3.1}$$

where $S_n^p := \sum_{k=1}^n X_k$. Let $(p_n)_{n \in \underline{N}} \subset \underline{N}$ be a sequence with $\lim_{n \to \infty} p_n = \infty$. In this section, we derive the following two complementary CLTs for $\underline{M}_q$-valued random variables $\Xi_n(\nu) := \Xi_n^{p_n}(\nu)$ under disjoint growth conditions for the dimensions p_n.

Theorem 3.1. *Assume that $\nu \in \underline{M}^1(\Pi_q)$ admits finite fourth moment.*
CLT I: *If $\lim_{n \to \infty} \frac{n}{p_n} = \infty$, then $\frac{\sqrt{p_n}}{n} \cdot \Xi_n(\nu)$ tends in distribution to the centered matrix variate normal distribution $\underline{N}(\mathbf{0}, T(\nu))$ with covariance matrix $T(\nu) := T_1(\nu) + T_2(\nu)$ where*

$$T_1(\nu)_{(i,j),(k,l)} = r_2(\nu)_{i,k} r_2(\nu)_{j,l} \quad and \quad T_2(\nu)_{(i,j),(k,l)} = r_2(\nu)_{i,l} r_2(\nu)_{j,k}. \tag{3.2}$$

CLT II: *If $\lim_{n \to \infty} \frac{n}{p_n} = c \in [0, \infty[$, then $\frac{1}{\sqrt{n}} \cdot \Xi_n(\nu)$ tends in distribution to the centered matrix variate normal distribution $\underline{N}(\mathbf{0}, \Sigma(\nu) + c T(\nu))$ (where $T(\nu)$ is given as in CLT I.)*

Notice that for $q = 1$ we obviously have $\nu \in \underline{M}^1([0, \infty[)$, $r_2(\nu) = \int_0^\infty x^2 d\nu(x)$, $T(\nu) = 2 r_2(\nu)^2$ and $\Sigma(\nu) = \int_0^\infty x^4 d\nu(x) - r_2(\nu)^2$. Therefore, Theorem 3.1 completely agrees with Theorem 1.1.

The proof of Theorem 3.1 will be divided into two main steps: In the first step we prove a reduced form of Theorem 3.1 assuming that ν has a compact support. In the second step we will show how to get rid of the support condition for ν. Both steps are based on the decomposition of $\Xi_n(\nu)$ via

$$\mathfrak{A}_n(\nu) := \sum_{i=1}^n A_i, \quad \text{with } A_i := \varphi_{p_n}(X_i)^2 - r_2(\nu), \tag{3.3}$$

$$\text{and} \quad \mathfrak{B}_n(\nu) := \sum_{i=1}^{p_n} B_i, \quad \text{with } B_i := \sum_{\alpha, \beta=1,\dots,n; \ \alpha \ne \beta} \left[X_\alpha^{(i,j)} X_\beta^{(i,l)}\right]_{1 \le j, l \le q}. \tag{3.4}$$

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We compute the covariance structure of $\mathfrak{A}_n(\nu)$ and $\mathfrak{B}_n(\nu)$ respectively: Since the random variables A_i $(i = 1, 2, \ldots)$ are independent and identically distributed, it is easily seen that

$$\mathbb{E}(A_k) = \mathbf{0}, \quad \mathbb{C}ov(A_i, A_j) = \delta_{i,j}\Sigma(\nu). \tag{3.5}$$

This gives

$$\frac{1}{n}\mathbb{C}ov(\mathfrak{A}_n(\nu)) = \frac{1}{n}\sum_{k=1}^{n}\mathbb{C}ov(A_k) = \Sigma(\nu). \tag{3.6}$$

By the independence of random variables X_k, $k \in \underline{\mathbb{N}}$ and Lemma 2.7 we obtain

$$\mathbb{E}(B_k) = \mathbf{0}, \quad \mathbb{C}ov(B_i, B_j) = \delta_{i,j}\frac{n(n-1)}{p_n^2}T(\nu). \tag{3.7}$$

We thus get

$$\lim_{n\to\infty}\frac{p_n}{n^2}\mathbb{C}ov(\mathfrak{B}_n(\nu)) = \lim_{n\to\infty}\frac{p_n}{n^2}\sum_{i=1}^{p_n}\mathbb{C}ov(B_i) = T(\nu). \tag{3.8}$$

In the following we will establish convergence in distribution of the random variables $\mathfrak{A}_n(\nu)$ and $\mathfrak{B}_n(\nu)$ (after appropriate scaling) by the method of moments [3, Theorem 30.2], which can be easily adapted to our general situation. As we are sure that the result is well-known, we omit the proof.

Theorem 3.2 (Method of moments). *Let $Y, Y_1, Y_2, \ldots$ be $\underline{M}_{p,q}$ valued random variables. Suppose that the distribution of Y is determined by its moments $M_k(Y)$ $(k \in \underline{N})$, that the Y_n have moments $M_k(Y_n)$ of all orders, and that*

$$\lim_{n\to\infty} M_k(Y_n) = M_k(Y)$$

for $k = 1, 2, \ldots$. Then the sequence $(Y_n)_n$ converges to Y in distribution.

Remark 3.3. Each matrix variate normal distribution $\underline{N}(M, \Sigma)$ on $\underline{M}_{p,q}$ or distribution with compact support are determined by its moments.

Definition 3.4. Let $(D_n)_{n\in\underline{N}}$, $(d_n)_{n\in\underline{N}}$ be a sequences of matrices from $\underline{M}_q$ and positive real numbers respectively. We write $D_n = O(d_n)$ as $n \to \infty$, if and only if $\|D_n\|_\infty = O(d_n)$ as $n \to \infty$.

Proposition 3.5. *Assume that $\nu \in \underline{M}^1(\Pi_q)$ has compact support. Then the asymptotic behaviour of $\mathfrak{A}_n := \mathfrak{A}_n(\nu)$ is given as follows:*

(a) If $\frac{n}{p_n} \to c \in [0, \infty[$ as $n \to \infty$, then $\frac{1}{\sqrt{n}}\mathfrak{A}_n$ tends in distribution to $\underline{N}(\mathbf{0}, \Sigma(\nu))$.

(b) If $\frac{n}{p_n} \to \infty$ as $n \to \infty$, then $\frac{\sqrt{p_n}}{n}\mathfrak{A}_n$ tends in distribution to $\delta_{\mathbf{0}}$.

Proof. If we prove that for all $k \in \underline{N}_0$, the k-th order moments

$$\frac{1}{n^{k/2}}E\left(\mathfrak{A}_n^{\otimes,k}\right) \quad \text{and} \quad \frac{p_n^{k/2}}{n^k}E\left(\mathfrak{A}_n^{\otimes,k}\right) \tag{3.9}$$

tend to the k-th order moment of the corresponding limit distribution in the case (a) and (b) respectively, the assertion follows by the method of moments 3.2. Therefore, we calculate

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(3.9) as $n \to \infty$. Since the random variables A_j are identically distributed, Theorem 2.2 shows that

$$\underline{\mathrm{E}}\left(\mathfrak{A}_n^{\otimes,k}\right) = \sum_{u=1}^{k} \sum_{\lambda \in C(k,u)} \binom{n}{u} \sum_{\pi \in \mathfrak{S}(\lambda)} \underline{\mathrm{E}}\pi(A_1, \ldots, A_u).$$

For $u \in \{1, \ldots, k\}$ and $\lambda \in C(k,u)$ we consider

$$T(\lambda) := \binom{n}{u} \sum_{\pi \in \mathfrak{S}(\lambda)} \underline{\mathrm{E}}\pi(A_1, \ldots, A_u) \in \underline{\mathrm{M}}_{q^k}. \tag{3.10}$$

If $\lambda_\alpha = 1$ for some α, i.e A_α appears exactly once in $\pi(A_1, \ldots, A_u)$, then each summand in (3.10) vanishes, which is due to the facts that $\underline{\mathrm{E}}(A_\alpha) = \mathbf{0} \in \underline{\mathrm{M}}_q$ and that the A_i are independent.

Suppose that $\lambda_\alpha \geq 2$ for each α and $\lambda_\alpha > 2$ for some α. Then $k > 2u$, and since $T(\lambda) = O(n^u)$ as $n \to \infty$, it follows that $(1/n^{k/2}) \cdot T(\lambda)$ and $(p_n^{k/2}/n^k) \cdot T(\lambda)$ in the cases $n/p_n \to c \in [0, \infty[$ and $n/p_n \to \infty$ respectively tend to zero as $n \to \infty$.

Now we turn to the case $\lambda = (2, \ldots, 2)$, in particular $k = 2u$. Let $Z_1, \ldots, Z_u$ be independent and $\underline{\mathrm{N}}(\mathbf{0}, \Sigma^2(\nu))$ distributed random variables. By Lemma 2.1, for any $\pi \in \mathfrak{S}(\lambda)$ there exist permutation matrices P_π and Q_π with

$$P_\pi \underline{\mathrm{E}}\big(\pi(A_1, \ldots, A_u)\big) Q_\pi = \underline{\mathrm{E}}\big(A_1 \otimes A_1 \otimes \ldots \otimes A_u \otimes A_u\big) = \Sigma(\nu) \otimes \ldots \otimes \Sigma(\nu)$$
$$= \underline{\mathrm{E}}(Z_1 \otimes Z_1 \otimes \ldots \otimes Z_u \otimes Z_u) = P_\pi \underline{\mathrm{E}}\big(\pi(Z_1, \ldots, Z_u)\big) Q_\pi,$$

and hence

$$\underline{\mathrm{E}}\pi(A_1, \ldots, A_u) = \underline{\mathrm{E}}\pi(Z_1, \ldots, Z_u) \quad \forall \ \pi \in \mathfrak{S}(\lambda).$$

Therefore, according to the Lemma 2.11 we have

$$T(\lambda) = \binom{n}{u} \sum_{\pi \in \mathfrak{S}(\lambda)} \underline{\mathrm{E}}\big(\pi(Z_1, \ldots, Z_u)\big) = \frac{n!}{(n-u)!} M_k(Z_1).$$

This proves that the moments in (3.9) converge to those of $\underline{\mathrm{N}}(\mathbf{0}, \Sigma^2(\nu))$ and the Dirac distribution $\delta_{\mathbf{0}}$ respectively. $\qquad \square$

Now we introduce some notation: Let k, $n \in \underline{\mathrm{N}}$ and $\underline{\mathrm{I}}_{k,n}$ the set of all $2k$-tuples $(i_1, j_1, \ldots, i_k, j_k)$ of positive integers less or equal n such that $i_\alpha \neq j_\alpha$ for all $\alpha = 1, \ldots, k$. For an $I \in \underline{\mathrm{I}}_{k,n}$ and $\pi = (\pi_1, \ldots, \pi_k) \in \underline{\mathrm{N}}^k$ we set

$$S(I, \pi) := \left[X_{i_1}^{(\pi_1, \alpha_1)} X_{j_1}^{(\pi_1, \beta_1)}\right]_{1 \leq \alpha_1, \beta_1 \leq q} \otimes \ldots \otimes \left[X_{i_k}^{(\pi_k, \alpha_k)} X_{j_k}^{(\pi_k, \beta_k)}\right]_{1 \leq \alpha_k, \beta_k \leq q}. \tag{3.11}$$

Each entry of $S(I, \pi) \in \underline{\mathrm{M}}_{q^k}$ is a product with k factors and corresponds to the tuple

$$\big((i_1, \pi_1, \alpha_1), (j_1, \pi_1, \beta_1), \ldots, (i_k, \pi_k, \alpha_k), (j_k, \pi_k, \beta_k)\big). \tag{3.12}$$

For (3.12) and two integers a, b we define

$$mult_{I,\pi}(a, b) = |\{\tau \in \{1, \ldots, k\} : (i_\tau, \pi_\tau, \alpha_\tau) = (a, b, \alpha_\tau) \text{ or } (j_\tau, \pi_\tau, \beta_\tau) = (a, b, \beta_\tau)\}|.$$

It is clear that $mult_{I,\pi}(a, b)$ does not depend on the indices α_τ and β_τ. Therefore, $mult_{I,\pi}(a, b)$ is the number of factors in an arbitrary entry of the matrix $S(I, \pi)$ which are coming from the b-th row of X_a. Moreover, we write $d(I)$ for the number of distinct elements in $\{I\}$. For an $m \in \{2, \ldots, 2k\}$ and $M \subsetneq \{1, \ldots, n\}$ with $|M| \leq k$ we consider following subsets of $\mathbb{I}_{k,n}$

$$\mathbb{J}_m := \left\{ I \in \mathbb{I}_{k,n} : \quad d(I) = m \right\},$$
$$\tilde{\mathbb{J}}_m := \{ I \in \mathbb{J}_m : \quad \{I\} = \{1, \ldots, m\} \},$$
$$\mathbb{J}_k^\exists(M) := \{ I \in \mathbb{J}_2 \cup \ldots \cup \mathbb{J}_k : \quad \{I\} \cap M \neq \emptyset \},$$
$$\mathbb{J}_k^\forall(M) := \{ I \in \mathbb{J}_2 \cup \ldots \cup \mathbb{J}_k : \quad m \in \{I\} \,\, \forall \,\, m \,\, \in \,\, M \},$$
$$\mathbb{J}^o(\pi) := \left\{ I \in \mathbb{I}_{k,n} : \quad \exists \,\, a, \,\, b \in \mathbb{N} : \quad mult_{I,\pi}(a, b) \text{ is odd} \right\}.$$

It is easily checked that for the cardinalities of $\mathbb{J}_m$, $\mathbb{J}_k^\exists(M)$ and $J_k^\forall(M)$ we have

$$|\mathbb{J}_m| \leq Cn^m, \quad \left|\mathbb{J}_k^\exists(M)\right| \leq Cn^{k-1}, \quad \left|\mathbb{J}_k^\forall(M)\right| \leq Cn^{k-|M|} \tag{3.13}$$

with some constant $C = C(k)$.

Proposition 3.6. *Assume that $\nu \in \mathbb{M}^1(\Pi_q)$ has compact support. Then the asymptotic behaviour of $\mathfrak{B}_n := \mathfrak{B}_n(\nu)$ is given as follows:*

(a) *If $\frac{n}{p_n} \to 0$ as $n \to \infty$, then $\frac{1}{\sqrt{n}}\mathfrak{B}_n$ tends in distribution to $\delta_{\mathbf{0}}$.*

(b) *If $\frac{n}{p_n} \to c \in]0, \infty]$ as $n \to \infty$, then $\left(\frac{\sqrt{p_n}}{n}\right)\mathfrak{B}_n$ tends in distribution to the normal distribution $\mathbb{N}(\mathbf{0}, T(\nu))$.*

Proof. According to the Theorem 3.2 it suffices to show that the k-th moments of $\frac{1}{\sqrt{n}}\mathfrak{B}_n$ and $\frac{\sqrt{p_n}}{n}\mathfrak{B}_n$ tend to the corresponding ones of the limiting distributions as $n \to \infty$. By using very similar arguments as in the proof of the Lemma 2.7 it is easily seen that B_i $(i = 1, 2, \ldots)$ are identically distributed. From this and Theorem 2.2 we conclude

$$\mathbb{E}\left(\mathfrak{B}_n^{\otimes,k}\right) = \sum_{v=1}^{k} \sum_{\mu \in C(k,v)} \binom{p_n}{v} \sum_{\pi \in \mathfrak{S}(\mu)} \mathbb{E}\pi(B_1, \ldots, B_v).$$

For an $v \in \{1, \ldots, k\}$, $\lambda \in C(k, v)$ and $\pi \in \mathfrak{S}(\mu)$ we consider $\pi(B_1, \ldots, B_v)$. The definition of B_a $(a \in M_v)$ in (3.4) enables us to write

$$\pi(B_1, \ldots, B_v) = B_{\pi_1} \otimes \ldots \otimes B_{\pi_k} = \sum_{I \in \mathbb{I}_{k,n}} S(I, \pi), \tag{3.14}$$

where each term $S(I, \pi)$ with $I = (i_1, j_1, \ldots, i_k, j_k)$ is given by (3.11). For a selected index $a \in M_n$, each entry of $S(I, \pi)$ may be regarded as a monomial in the variables X_a (that is, in $X_a^{(\alpha, \beta)}$ with $\alpha, \beta \in \mathbb{N}$) while the random variables coming from other indices are considered as constant. In this view, for any $I \in \mathbb{J}^o(\pi)$, each entry of $S(I, \pi)$ is for some $a \in \{1, \ldots, n\}$ and $b \in \{1, \ldots, v\}$ a monomial in the variable X_a which is not even in row b. And hence Theorem 2.10 clearly forces

$$\mathbb{E}(S(I, \pi)) = \mathbf{0} \quad \forall \,\, I \in \mathbb{J}^o(\pi). \tag{3.15}$$

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Therefore, since $\mathrm{J}_m \subset \mathrm{J}^o(\pi)$ for $m > k$, we conclude from (3.14) that

$$\underline{\mathrm{E}}\pi(B_1, \ldots, B_v) = \sum_{m=2}^{k} \sum_{I \in \mathrm{J}_m} \underline{\mathrm{E}}S(I, \pi). \tag{3.16}$$

By the definition of $S(I, \pi)$ in (3.11) and Theorem 2.10, the terms in the last sum are uniformly bounded by $C \cdot n^m$ with a constant $C > 0$, that is,

$$\sup_{I \in \mathrm{J}_m} \|ES(I, \pi)\|_\infty = O(n^m).$$

Moreover, according to (3.13) we have $|\mathrm{J}_m| \leq Cn^m$ for a constant $C > 0$, and hence we get

$$E\pi(B_1, \ldots, B_v) = \sum_{I \in \mathrm{J}_k} \underline{\mathrm{E}}S(I, \pi) + O\left(\frac{n^{k-1}}{p_n^k}\right) = O\left(\frac{n^k}{p_n^k}\right). \tag{3.17}$$

For $v \in \{1, \ldots, k\}$ and $\mu \in C(k, v)$ let us consider

$$T(\mu) := \binom{p_n}{v} \sum_{\pi \in \mathfrak{S}(\mu)} \underline{\mathrm{E}}\pi(B_1, \ldots, B_v). \tag{3.18}$$

If $\mu_\alpha = 1$ for some α, i.e for any $\pi \in \mathfrak{S}(\mu)$ the factor B_α appears exactly once in the product $\pi(B_1, \ldots, B_v)$, and therefore, each $I \in \mathrm{J}_{k,n}$ from the Representation (3.14) of $\pi(B_1, \ldots, B_v)$ is necessarily from $\mathrm{J}^o(\pi)$, and hence (3.15) gives $T(\mu) = 0$.

Suppose that $\mu_\alpha \geq 2$ for each α and $\mu_\alpha > 2$ for some α, that is, in particular $k > 2v$. From (3.17) we conclude that $n^{-k/2}T(\mu) = O(n^{k/2}p_n^{v-k})$ and $p_n^{k/2}n^{-k}T(\mu) = O(p_n^{v-k/2})$ tend to $\mathbf{0}$ as $n \to \infty$ in the case (a) $\frac{n}{p_n} \to 0$ and case (b) $\frac{p_n}{n} \to 0$ respectively.

We now turn to the case $\mu = (2, \ldots, 2)$, in particular $k = 2v$. By Eq. (3.17) it follows in the case (a) that $n^{-v}T(\mu) = O((n/p_n)^{k-v})$ and hence that $n^{-v}T(\mu)$ converges to zero as $n \to \infty$.

Since $X_1, X_2, \ldots$ are i.i.d., we have

$$\sum_{I \in \mathrm{J}_k} \underline{\mathrm{E}}S(I, \pi) = \binom{n}{k} \sum_{I \in \tilde{\mathrm{J}}_k} \underline{\mathrm{E}}S(I, \pi).$$

Therefore, by using Eq. (3.17),

$$T(\mu) = \frac{p_n!}{(p_n - v)!} \frac{n!}{(n - k)!} \frac{1}{v!} \sum_{\pi \in \mathfrak{S}(\mu)} \frac{1}{p_n^k} \frac{p_n^k}{k!} \sum_{I \in \tilde{\mathrm{J}}_k} \underline{\mathrm{E}}S(I, \pi) + O\left(\frac{n^{k-1}}{p_n^{k-v}}\right).$$

Let $Z_1, \ldots, Z_v$ be independent and $\mathrm{N}(\mathbf{0}, T(\nu))$ distributed random variables. By Lemma 3.7, which is proven below, we obtain

$$\lim_{n \to \infty} \frac{p_n^v}{n^k} T(\mu) = \frac{1}{v!} \sum_{\pi \in \mathfrak{S}(\mu)} \underline{\mathrm{E}}\pi(Z_1, \ldots, Z_v).$$

The required result then follows from Lemma 2.11 and Method of moments 3.2. $\qquad\square$

Lemma 3.7. *Let* $v \in \underline{N}$, $k = 2v$, $\mu = (2, \ldots, 2) \in C(k, v)$, $\pi \in \mathfrak{S}(\mu)$ *and* $Z_1, \ldots, Z_v$ *be independent* $\underline{N}(\mathbf{0}, T(\nu))$ *distributed random variables. Then*

$$\underline{E}\pi(Z_1, \ldots, Z_v) = \frac{p_n^k}{k!} \sum_{I \in \tilde{J}_k} \underline{E}S(I, \pi) =: R(\pi).$$

Proof. According to the Lemma 2.1 there is no loss of generality in assuming that $\pi = (1, 1, 2, 2, \ldots, v, v)$. We set

$$J_{k,\pi} := \left\{ (i_1, j_1, \ldots, i_k, j_k) \in \tilde{J}_k : \quad \{i_\alpha, j_\alpha\} = \{i_\beta, j_\beta\} \text{ if } \pi_\alpha = \pi_\beta \right\}.$$

It is easy to check that $\tilde{J}_k \setminus J_{k,\pi} \subset J^o(\pi)$. Therefore, by Eq. (3.15),

$$\sum_{I \in \tilde{J}_k} \underline{E}S(I, \pi) = \sum_{I \in J_{k,\pi}} \underline{E}S(I, \pi).$$

For a permutation $\sigma \in S_k := Sym(\{1, \ldots, k\})$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_u) \in \underline{Z}_2^v$ we consider the functions

$$\varphi_\sigma : J_{k,\pi} \longrightarrow J_{k,\pi}, \qquad (i_1, j_1, \ldots, i_k, j_k) \mapsto (\sigma(i_1), \sigma(j_1), \ldots, \sigma(i_k), \sigma(j_k))$$

$$\theta_\varepsilon : J_{k,\pi} \longrightarrow J_{k,\pi}, \qquad (i_1, j_1, \ldots, i_k, j_k) \mapsto (r_1, t_1, \ldots, r_k, t_k),$$

where $(r_1, t_1, \ldots, r_k, t_k)$ is defined as follows: for any $\alpha, \beta \in M_k$ with $\alpha < \beta$ and $\pi_\alpha = \pi_\beta \in \{1, \ldots, v\}$ we have

$$(r_\alpha, t_\alpha, r_\beta, t_\beta) = \begin{cases} (i_\alpha, j_\alpha, i_\beta, j_\beta), & \text{if } \varepsilon_{\pi_\alpha} = 0, \\ (i_\alpha, j_\alpha, j_\beta, i_\beta), & \text{if } \varepsilon_{\pi_\alpha} = 1. \end{cases}$$

It is easily seen that φ_σ and θ_ε are well defined. Let $I_0 := (1, 2, 1, 2, \ldots, k-1, k, k-1, k) \in J_{k,\pi}$. By standard verification we obtain a one-to-one correspondence between $S_k \times \underline{Z}_2^v$ and $J_{k,\pi}$ via the map $\Psi : (\sigma, \varepsilon) \mapsto \varphi_\sigma(\theta_\varepsilon(I_0))$. Since $X_1, X_2, \ldots$ are independent identically distributed we have for all $\sigma \in S_k$

$$\underline{E}S(\varphi_\sigma(I), \pi) = ES(I, \pi) \quad \forall\, I \in J_{k,\pi}. \tag{3.19}$$

For an $\varepsilon \in \underline{Z}_2$ we consider the algebraic operation

$$\varepsilon(a, b) = \begin{cases} a, & \text{if } \varepsilon = 0, \\ b, & \text{if } \varepsilon = 1. \end{cases}$$

By Equation (3.19) it follows that

$$R(\pi) = \frac{1}{k!} \sum_{(\sigma, \varepsilon) \in S_k \times \underline{Z}_2^v} p_n^k \underline{E}S(\Psi(\sigma, \varepsilon), \pi) = \sum_{\varepsilon \in \underline{Z}_2^v} p_n^k \underline{E}S(\Psi(id, \varepsilon), \pi)$$

$$= \sum_{\varepsilon \in \underline{Z}_2^v} \varepsilon_1(T_1, T_2) \otimes \ldots \otimes \varepsilon_v(T_1, T_2) = \underline{E}\pi(Z_1, \ldots, Z_v),$$

where $T_1 := T_1(\nu)$ and $T_2 := T_2(\nu)$ are defined as in (3.2). $\square$

Now, in order to prove Theorem 3.1 for sequences p_n with $p_n/n \to c \in]0, \infty[$ we show that $\mathfrak{A}_n := \mathfrak{A}_n(\nu)$ and $\mathfrak{B}_n := \mathfrak{B}_n(\nu)$ are asymptotically independent.

Proposition 3.8. *Assume that $\nu \in M^1(\Pi_q)$ has compact support and that $\lim_{n\to\infty} \frac{n}{p_n} =: c \in {]0, \infty[}$. Then the random variables $\mathfrak{A}_n$ and $\mathfrak{B}_n$ are asymptotically independent, that is, for all $0 \le l \le k$ and all $\sigma \in \mathfrak{S}((l, k-l))$*

$$F(n; \sigma) := \frac{1}{n^{k/2}} \left[\underline{E}\sigma(\mathfrak{A}_n, \mathbb{1}) \circ \underline{E}\sigma(\mathbb{1}, \mathfrak{B}_n) - \underline{E}\sigma(\mathfrak{A}_n, \mathfrak{B}_n) \right]$$

tends to zero as $n \to \infty$.

Proof. According to the Lemma 2.1 there is no loss of generality in assuming that $\sigma = (1, \dots 1, 2, \dots, 2)$. From Theorem 2.2, by using symmetry argument, we conclude

$$F(n; \sigma) = \frac{1}{n^{k/2}} \left(\underline{E}\left(\mathfrak{A}_n^{\otimes, l}\right) \otimes \underline{E}\left(\mathfrak{B}_n^{\otimes, k-l}\right) - \underline{E}\left(\mathfrak{A}_n^{\otimes, l} \otimes \mathfrak{B}_n^{\otimes, k-l}\right) \right)$$

$$= \frac{1}{n^{k/2}} \sum_{u=1}^{l} \sum_{v=1}^{k-l} \sum_{\lambda \in C(l,u)} \sum_{\mu \in C(k-l,v)} \binom{n}{u} \binom{p_n}{v} \sum_{\pi \in \mathfrak{S}(\lambda)} \sum_{\pi' \in \mathfrak{S}(\mu)} H(\pi, \pi'),$$

with

$$H(\pi, \pi') = \underline{E}\pi(A_1, \dots, A_u) \otimes \underline{E}\pi'(B_1, \dots, B_v) - \underline{E}\left(\pi(A_1, \dots, A_u) \otimes \pi'(B_1, \dots, B_v)\right).$$

If $\mu_\alpha = 1$ for some $\alpha \in \{1, \dots, v\}$, then each entry of $\pi'(B_1, \dots, B_v)$ is not an even polynomial and thus so is $\pi(A_1, \dots, A_u) \otimes \pi'(B_1, \dots, B_v)$ neither. Therefore, $H(\pi, \pi') = 0$ by Theorem 2.10.

Suppose that $\mu_\alpha \ge 2$ for each α. By Eq. (3.16) we have

$$H(\pi, \pi') = \sum_{I \in J_2 \cup \dots \cup J_{k-l}} \left(\underline{E}\pi(A_1, \dots, A_u) \otimes \underline{E}S(I, \pi') - \underline{E}\left(\pi(A_1, \dots, A_u) \otimes S(I, \pi')\right) \right).$$

(3.20)

Let $M := \{1, \dots, u\}$ and $G := \{\alpha \in M : \lambda_\alpha = 1\}$. We consider the I-th term in the sum above, which will be denoted by $T(I)$. Is $I \notin J_{k-l}^{\exists}(M)$, that is, $\{I\} \cap M = \emptyset$, and thus $A_1, \dots, A_u$ are independent from $S(I, \pi')$. This clearly forces $T(I) = 0$. Is $I \notin J_{k-l}^{\forall}(G)$, that is, there exists $\tau \in G$ with $\tau \notin \{I\}$, and therefore, A_τ is independent from A_i ($i \in M \setminus \{\tau\}$) and $S(I, \pi')$. We thus get $T(I) = 0$ from (3.5).

Taking (3.13) into account, we see that the number of nonzero summands in (3.20) is bounded above $\min(n^{k-l-1}, n^{k-l-|G|})$. On the other side, Lemma 2.10 yields that each of them is bounded above C/p_n^{k-l} where $C > 0$ is a suitable global constant. Summarized we get

$$\|H(\pi, \pi')\| \le C \cdot \min(n^{-1}, n^{-|G|}).$$

(3.21)

Since $\mu \in C(k-l, v)$ with $\mu_\alpha \ge 2$ for all $\alpha \in \{1, \cdots, v\}$ we have that $k - l \ge 2v$. Moreover, since $\lambda \in C(l, u)$ we get $l \ge 2u - |G|$. And hence, by straightforward calculation using $n/p_n \to c \in {]0, \infty[}$ we conclude from (3.21) that for suitable constants C_i,

$$\|F(n, \sigma)\| \le \frac{C_1}{n^{k/2}} \sum_{u=1}^{l} \sum_{v=1}^{k-l} \sum_{\lambda \in C(l,u)} \sum_{\mu \in C(k-l,v)} \binom{n}{u} \binom{p_n}{v} \min(n^{-1}, n^{-|G|})$$

$$\le \frac{C_2}{n^{k/2}} \sum_{u=1}^{l} \sum_{v=1}^{k-l} \sum_{\lambda \in C(l,u)} n^{u+v} \min(n^{-1}, n^{-|G|}) \le \frac{C_3}{\sqrt{n}}.$$

This completes the proof. $\qquad\square$

Proof of Theorem 3.1 for $\nu \in \underline{M}^1(\Pi_q)$ with compact support.
If $n/p_n \to \infty$ then $\frac{\sqrt{p_n}}{n}\mathfrak{A}_n \xrightarrow{d} \delta_{\mathbf{0}}$ and $\frac{\sqrt{p_n}}{n}\mathfrak{B}_n \xrightarrow{d} \underline{N}(\mathbf{0}, T(\nu))$ according to Propositions 3.5 and 3.6. This clearly forces $\frac{\sqrt{p_n}}{n}\Xi_n(\nu) \xrightarrow{d} \underline{N}(\mathbf{0}, T(\nu))$ by Slutsky's Theorem. Suppose that $n/p_n \to 0$. Then we get as above $\frac{1}{\sqrt{n}}\Xi_n(\nu) \xrightarrow{d} \underline{N}(\mathbf{0}, \Sigma(\nu))$. It remains only to check the convergence in the case $n/p_n \to c \in]0, \infty[$. Let $k \in \underline{N}$. By Theorem 2.2,

$$M_k(\Xi_n(\nu)) = \underline{E}\left((\mathfrak{A}_n + \mathfrak{B}_n)^{\otimes k}\right) = \sum_{l=0}^{k} \sum_{\pi \in \mathfrak{S}((l,k-l))} \underline{E}\pi(\mathfrak{A}_n, \mathfrak{B}_n).$$

Therefore, by Proposition 3.8,

$$\lim_{n \to \infty} M_k\left(\frac{1}{\sqrt{n}}\Xi_n(\nu)\right) = \lim_{n \to \infty} \frac{1}{n^{k/2}} \sum_{l=0}^{k} \sum_{\pi \in \mathfrak{S}((l,k-l))} \underline{E}\pi(\mathfrak{A}_n, \mathbb{1}) \circ \underline{E}\pi(\mathbb{1}, \mathfrak{B}_n).$$

Consider independent random variables Z_1, Z_2 and Z with distributions $\underline{N}(\mathbf{0}, \Sigma(\nu))$, $\underline{N}(\mathbf{0}, cT(\nu))$ and $\underline{N}(\mathbf{0}, \Sigma(\nu) + cT(\nu))$ respectively. Propositions 3.5, 3.6 and Lemma 2.12 now lead to

$$\lim_{n \to \infty} M_k(\Xi_n(\nu)) = \sum_{l=0}^{k} \sum_{\pi \in \mathfrak{S}((l,k-l))} \underline{E}\pi(Z_1, \mathbb{1}) \circ \underline{E}\pi(\mathbb{1}, Z_2) = M_k(Z).$$

$\square$

In order to get rid of the assumption that $supp(\nu)$ is compact, we introduce for an $a > 0$ the truncated $\underline{M}_{p_n,q}$-valued random variables

$$X_{k,a} := \begin{cases} X_k, & \text{if } \|\varphi_{p_n}(X_k)\| \leq a, \\ \mathbf{0}, & \text{otherwise} \end{cases} \quad k = 1, 2, \ldots$$

Let us denote by ν_a the distribution of $\varphi_{p_n}(X_{1,a})$ (which is not dependent on p_n). Obviously, the sequence $X_{k,a}$, $k \in \underline{N}$, are i.i.d. with the radial law $\nu_{p_n,a} \in \underline{M}(\underline{M}_{p_n,q})$ which corresponds to ν_a. We define $\Xi_n(\nu_a)$, $\mathfrak{A}_n(\nu_a)$, $A_{j,a}$ $(j = 1, \ldots, n)$, $\mathfrak{B}_n(\nu_a)$ and $B_{j,a}$ $(j = 1, \ldots, p_n)$ according to (3.1), (3.3) and (3.4) respectively, by taking $X_{k,a}$ instead of X_k, $k \in \underline{N}$. Clearly, we have $\Xi_n(\nu_a) = \mathfrak{A}_n(\nu_a) + \mathfrak{B}_n(\nu_a)$.

In the following we show that $\Xi_n(\nu_a)$ is a "good" approximation of $\Xi_n(\nu)$. To formulate this exactly, we first fix some $\delta > 0$ and a sequence $(p_n)_n$; we then introduce the sequence $(\delta_n)_n$ by

$$\delta_n := \begin{cases} \delta \cdot \sqrt{n}, & \text{if } \frac{n}{p_n} \to c \in [0, \infty[, \\ \delta \cdot \frac{n}{\sqrt{p_n}}, & \text{if } \frac{n}{p_n} \to \infty. \end{cases} \tag{3.22}$$

In the next lemmas we show that the events

$$\{\|\mathfrak{A}_n(\nu_a) - \mathfrak{A}_n(\nu)\| > \delta_n\} \text{ and } \{\|\mathfrak{B}_n(\nu_a) - \mathfrak{B}_n(\nu)\| > \delta_n\}$$

have arbitrary small probabilities for an a and n large enough.

Lemma 3.9. *For all $\varepsilon > 0$, $\delta > 0$ there exist a_0, $n_0 \in \underline{N}$ such that for all n, $a \in \underline{N}$ with $a \geq a_0$ and $n \geq n_0$*

$$\underline{P}\left(\|\mathfrak{A}_n(\nu) - \mathfrak{A}_n(\nu_a)\| > \delta_n\right) \leq \varepsilon.$$

Proof. Let $\delta > 0$ and $(\delta_n)_n$ be a sequence as in (3.22). Since $(A_i - A_{i,a})$, $(i = 1, 2, \dots)$ are i.i.d., it follows by Chebychev inequality that

$$\underline{P}\left(\|\mathfrak{A}_n(\nu) - \mathfrak{A}_n(\nu_a)\| \geq \delta_n\right) \leq \frac{n}{\delta_n^2} \underline{E}\left(\|A_1 - A_{1,a}\|^2\right). \tag{3.23}$$

Using triangle inequality we obtain

$$\sup_{a \in \underline{N}} \|A_{1,a}\|^2 \leq \left(\left\|\varphi_{p_n}^2(X_1)\right\| + \|r_2(\nu)\|\right)^2 \in L^1(\Omega),$$

Therefore, the set $\{\|A_{1,a}\|^2 : a \in \underline{N}\}$ is uniformly integrable. On the other side, since the random variable $\|A_1\|$ is almost surely finite, $\|A_{1,a}\|^2$ converges a.s. to $\|A_1\|^2$ as $a \to \infty$. We thus get

$$\|A_{1,a}\|^2 \longrightarrow \|A_1\|^2 \quad \text{in } L^1. \tag{3.24}$$

By taking (3.23) and (3.24) into account, the lemma follows. $\qquad\square$

Lemma 3.10. *For all $\varepsilon > 0$, $\delta > 0$ there exist a_0, $n_0 \in \underline{N}$ such that for all n, $a \in \underline{N}$ with $a \geq a_0$ and $n \geq n_0$*

$$\underline{P}\left(\|\mathfrak{B}_n(\nu) - \mathfrak{B}_n(\nu_a)\| > \delta_n\right) \leq \varepsilon.$$

Proof. Let $\delta > 0$ and $(\delta_n)_n$ be a sequence as in (3.22). By Chebychev inequality it follows that

$$\underline{P}\left(\|\mathfrak{B}_n(\nu) - \mathfrak{B}_n(\nu_a)\| \geq \delta_n\right) \leq \frac{1}{\delta_n^2} \sum_{j,i=1}^{p_n} \underline{E}\left(\langle B_i - B_{i,a}, B_j - B_{j,a}\rangle\right). \tag{3.25}$$

Using Lemma 2.7 one can easily compute that

$$\underline{E}\left(\langle B_i, B_j\rangle\right) = \delta_{ij} \cdot \frac{n(n-1)}{p_n^2} \sum_{l,k=1}^{q} r_2(\nu)_{l,l} r_2(\nu)_{k,k} + r_2(\nu)_{l,k} r_2(\nu)_{l,k}$$

$$\underline{E}\left(\langle B_{i,a}, B_{j,a}\rangle\right) = \delta_{ij} \cdot \frac{n(n-1)}{p_n^2} \sum_{l,k=1}^{q} r_2(\nu_a)_{l,l} r_2(\nu_a)_{k,k} + r_2(\nu_a)_{l,k} r_2(\nu_a)_{l,k}$$

With the notation

$$\tilde{r}_2(a; n) := \left(\underline{E}\left(X_{1,a}^{(1,l)} X_1^{(1,k)}\right)\right)_{1 \leq l,k \leq q}$$

we see at once that

$$E\left(\langle B_i, B_{j,a}\rangle\right) = \delta_{ij} n(n-1) \sum_{l,k=1}^{q} \tilde{r}_2(a; n)_{l,l} \tilde{r}_2(a; n)_{k,k} + \tilde{r}_2(a; n)_{l,k} \tilde{r}_2(n; a)_{l,k}.$$

For l, $k \in \{1, \dots, q\}$ we obtain

$$\tilde{r}_2(a; n)_{l,k} = \frac{1}{p_n} r_2(\nu)_{l,k} - \int_{\{\|X_1\| > a\}} X_1^{(1,l)} X_1^{(1,k)} d\underline{P}. \tag{3.26}$$

By Cauchy-Schwarz inequality and straightforward calculation we get

$$0 \leq \left|\int_{\{\|X_1\| > a\}} X_1^{(1,l)} X_1^{(1,k)} d\underline{P}\right| \leq \frac{c}{a p_n}, \quad a \to \infty$$

uniformly in n with some constant $c > 0$. From this and (3.26) we deduce

$$p_n \tilde{r}_2(a; n) = r_2(\nu) + O\left(\frac{1}{a}\right)$$

and hence

$$\forall\, \varepsilon > 0 \;\exists\, M > 0 \;\forall\, n \geq M,\; \forall\, a \geq M: \quad 0 \leq \frac{p_n^2}{n^2} \mathrm{E}\left(\|B_i - B_{i,a}\|\right) \leq \varepsilon.$$

Finally, this and (3.25) lead to the claim. $\qquad\square$

Corollary 3.11. *For all $\varepsilon > 0$, $\delta > 0$ there exist a_0, $n_0 \in \underline{N}$ such that for all n, $a \in \underline{N}$ with $a \geq a_0$ and $n \geq n_0$*

$$\underline{P}(\|\Xi_n(\nu) - \Xi_n(\nu_a)\| > \delta_n) \leq \varepsilon,$$

where $\delta_n = \delta\sqrt{n}$ if $n/p_n \to c \in [0, \infty[$ and $\delta_n = \delta\frac{n}{\sqrt{p_n}}$ if $n/p_n \to \infty$.

Proof. For an $\delta > 0$ we observe

$$\mathrm{P}\left(\|\Xi_n(\nu) - \Xi_n(\nu_a)\| > \delta_n\right) \leq \underline{P}\left(\|\mathfrak{A}_n - \mathfrak{A}_{n,a}\| > \frac{\delta_n}{2}\right) + \mathrm{P}\left(\|\mathfrak{B}_n - \mathfrak{B}_{n,a}\| > \frac{\delta_n}{2}\right).$$

Combining this with Lemmas 3.9 and 3.10, the corollary follows. $\qquad\square$

Proof of Theorem 3.1.
Let us first prove the CLT I. In this case the normalisation is given by $\frac{\sqrt{p_n}}{n}$ and for the growth of p_n we have the condition $n/p_n \to \infty$ as $n \to \infty$. We set $\xi_n := \frac{\sqrt{p_n}}{n}\Xi_n(\nu)$ and $\xi_{n,a} = \frac{\sqrt{p_n}}{n}\Xi_n(\nu_a)$ and denote their distributions by μ_n and $\mu_{n,a}$ respectively. Moreover, we write τ_ν instead of $\underline{N}(\mathbf{0}, T(\nu))$. Using triangle inequality, we deduce that

$$\left|\int f d\mu_n - \int f d\tau_\nu\right| \leq \left|\int f d\mu_n - \int f d\mu_{n,a}\right| + \tag{3.27}$$
$$+ \left|\int f d\mu_{n,a} - \int f d\tau_{\nu_a}\right| + \left|\int f d\tau_{\nu_a} - \int f d\tau_\nu\right|.$$

Let $\varepsilon > 0$, $f \in \mathrm{C}_b^u(\Pi_q)$ be a bounded uniformly continuous function on Π_q and $A_\delta := \{\|\xi_n - \xi_{n,a}\| \leq \delta\}$ ($\delta > 0$). It follows that

$$\exists\, \delta > 0: \quad \int_{A_\delta} |f \circ \xi_n - f \circ \xi_{n,a}| \, d\mathrm{P} \leq \varepsilon.$$

On the other hand, by Corollary 3.11,

$$\exists\, a_0,\; n_0 > 0: \quad \int_{\Omega \setminus A_\delta} |f \circ \xi_n - f \circ \xi_{n,a}| \, d\mathrm{P} \leq 2\varepsilon \, \|f\|_\infty \quad \forall\, a \geq a_0,\; n \geq n_0.$$

This gives us the following estimation for the first summand in (3.27):

$$\exists\, a_0,\; n_0 > 0: \; \left|\int f d\mu_n - \int f d\mu_{n,a}\right| \leq \varepsilon(1 + 2\|f\|_\infty) \quad \forall\, a \geq a_0,\; n \geq n_0. \tag{3.28}$$

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Since ν_a has a compact support, we conclude from 3.1 that $\mu_{n,a}$ weakly converges to τ_{ν_a} ($a > 0$), hence that

$$\forall\, a > 0\,\, \exists\, n_0 > 0 : \left| \int f d\mu_{n,a} - \int f d\tau_{\nu_a} \right| \leq \varepsilon \quad \forall\, n \geq n_0. \tag{3.29}$$

Finally, it is evident that

$$\exists\, a_0 > 0 : \left| \int f d\tau_{\nu_a} - \int f d\tau_\nu \right| \leq \varepsilon \quad \forall\, a \geq a_0. \tag{3.30}$$

Taking (3.28), (3.29) and (3.30) into account, we obtain

$$\exists\, n_0 > 0 : \left| \int f d\mu_n - \int f d\tau_\nu \right| \leq \varepsilon(3 + 2\,\|f\|_\infty) \quad \forall\, n \geq n_0,$$

which completes the proof of CLT I in Theorem 3.1. The same proof works for CLT II. $\qquad\square$

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